

Uncertainty and UncertaintyGUM *Mathematica* Functions

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Abstract

Two functions from the authors *Mathematica* package are demonstrated. `UncertaintyGUM` is based on the Guide to the Expression of Uncertainty in Measurement (GUM) [1]. The more powerful `Uncertainty` use the propagation of distributions to find analytically, numerically, or statistically the expectation \pm standard deviation for expression of random variables with prescribed statistical distributions. These two functions gives the same results for the linear expression $c_1*x_1+c_2*x_2+c_3*x_3$ with normal or multinormal statistical distributions of variables x_1 , x_2 , and x_3 . The approximate results of `UncertaintyGUM` and exact one of `Uncertainty`, for the nonlinear expressions `Sin[x]` and x^n of the normally distributed random variable x , are compared in three interactive demonstrations. This paper is a *Mathematica* notebook transformed to the new CDF (Computable Document Format) supported by the free CDF Player distributed by the Wolfram Research [3].

1. Introduction

The Guide to the Expression of Uncertainty in Measurement (GUM) [1], and its Supplement 1 [2], are internationally accepted master documents for the evaluation of uncertainty. GUM and its Supplement stimulated the appearance of numerous papers and calculators for uncertainty. The Metrology journal published many important articles, describing interesting details and improvements of GUM and its Supplement.

The Guide provides a framework for assessing uncertainty, based on the law of propagation of uncertainty. However the GUM approach is exact only for linear models, since it is based on the Taylor expansion neglecting higher order terms. Thus for nonlinear models the results could be quite misleading. To avoid these limitations a Supplement 1 [2] considers uncertainty evaluation by using the propagation of distributions and recommends Monte-Carlo method for implementation.

During the last years the authors developed *Mathematica* package supporting uncertainty computations. This paper demonstrate only 2 of the developed functions `UncertaintyGUM` and `Uncertainty`. The latter is more powerful because it could find analytically, numerically, or statistically the expectation \pm standard deviation for expression of variables with prescribed statistical distributions.

Examples for both functions are solved by using the simplified versions of our uncertainty calculator without keyboard. It is able to evaluate only expressions included in the pop-up menu `input`. For easy reading and transforming the paper in pdf format, the simplified calculator is pasted at many place to show formulas.

This document is a *Mathematica* notebook transformed to the new Computable Document Format (CDF). It could be evaluated by the free CDF Player distributed by the Wolfram Research [3]. Click on the number in square brackets to jump at the corresponding reference. All *Mathematica* functions in this text are written in `Courier` font family.

To change the text size of this notebook click on the fillet up triangle \blacktriangle at the right down corner and select one of

the marks 50%, 75%, 100%, 125%, 150%, 200%, or 300%. To change the notebook size click on the middle button at the right upper corner to appears a single square. Than locate the mouse at the right bottom corner, and when an arrow appears press the left mouse button and move to the desired location.

2. Simplified calculator

The full keyboard version of our uncertainty calculator will be given in another paper. The simplified version of this calculator evaluate only expressions included in the pop-up menu input. All examples bellow are solved using the simplified calculator. In this section it is empty.

Click on the light blue rectangle to open input pop-up menu and select one of its elements. The corresponding *Mathematica* input line appears on the calculator screen. Than click on the *enter* key. The input line becomes bold and the result appears on the corresponding output line. Thus you could obtain all results given bellow.

Click on the *clear all* key or click on \oplus at the right upper corner and select *Initial Setting* to clear the screen.

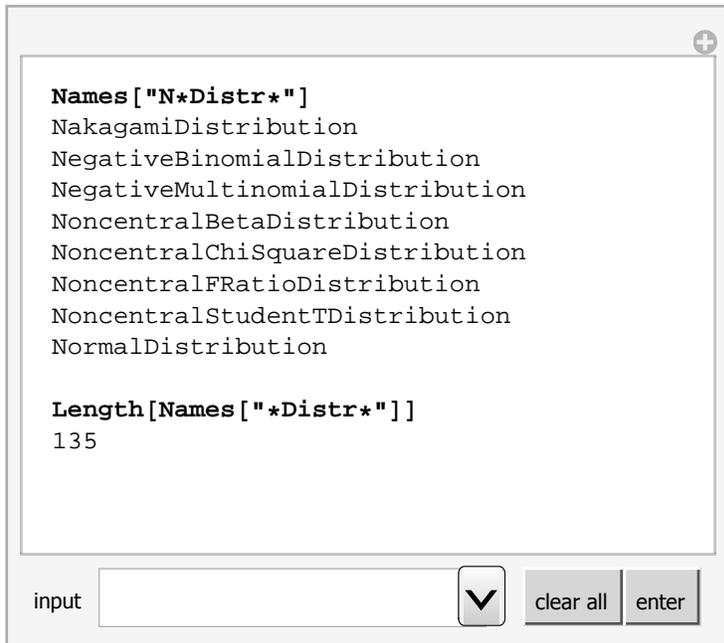


2. Statistical Distributions

Almost all *Mathematica* built-in objects are full English names starting with capital letters. For example the `NormalDistribution[μ , σ]` represent a Gaussian normal distribution with mean μ and standard deviation σ . *Mathematica* support 135 statistical distributions.

Derived distributions are modifications to existing distributions. There is a variety of ways in which you can arrive at modified distributions, including functions of random variables, weighted mixtures of distributions, truncated or censored distributions, marginals from higher-dimensional distributions, or joining marginals to a dependency kernel, as in copulas. Derived distributions behave just like any other distribution in *Mathematica*.

The input line `Names["N*Distr*"]` gives a list of distributions which match the string "N*Distr*". `Length[Names["*Distr*"]]` gives the number of the built-in statistical distributions.



```
Names["N*Distr*"]
NakagamiDistribution
NegativeBinomialDistribution
NegativeMultinomialDistribution
NoncentralBetaDistribution
NoncentralChiSquareDistribution
NoncentralFRatioDistribution
NoncentralStudentTDistribution
NormalDistribution

Length[Names["*Distr*"]]
135
```

input

3. UncertaintyGUM, Uncertainty, and powerXn functions

The documentation of our UncertaintyGUM, Uncertainty, and powerXn functions are given bellow.

?UncertaintyGUM
 UncertaintyGUM[f[x], x == $\mu \pm \sigma$, case] or
 UncertaintyGUM[f[x₁, ..., x_n], {x₁ == $\mu_1 \pm \sigma_1$, ..., x_n == $\mu_n \pm \sigma_n$ }, case]
 gives first order series approximation
 of f (Expectation ± StandardDeviation).
 case = 1 or IdentityMatrix[n] (uncorrelated variables),
 case = 2 nonlinear f (more series terms included in case 1),
 case = 3 or n×n constant matrix r_{i,j}=1 (fully correlated variables).
 case = n×n matrix: r_{i,j} = r_{j,i}, r_{i,i} = 1, -1 ≤ r_{i,j} ≤ 1.

?Uncertainty
 Uncertainty[expr, case, n] gives Expectation ± StandardDeviation of expr.
 The expr is an expression of the random variables x or x₁, ..., x_n.
 case = x ≈ dist assumes that x follows the probability distribution dist.
 case = x ≈ data assumes that x follows
 the probability distribution given by data.
 case = {x₁ ≈ dist₁, ..., x_n ≈ dist_n} assumes that x₁, ..., x_n are
 independent and follows the distributions dist₁, ..., dist_n.
 case = {x₁, ..., x_n} ≈ dist assumes that {x₁, ..., x_n}
 follows the multivariate distribution dist.
 The probability distribution dist can be any
 symbolic probability distribution.
 The argument n is optional. The n>0 specify
 the number of RandomVariate used.

?powerXn
 powerXn[μ, σ, n] implement the solution of Uncertainty[x^n,
 x≈NormalDistribution[μ, σ]], n ∈ Integers]

input

The results given by Uncertainty function could be classified as:

a) *symbolic*, when n is missing and symbols are used. If such solutions exist they cover a large class of expressions, but the time used for the derivation could be long.

b) *exact numeric*, when n is missing and exact numbers (integers or rationals) are used in the input. The obtained results could be transformed in numeric with any-digit precision.

c) *numeric*, when n is missing and real numbers are used in the input.

d) *statistical*, when n is positive, RandomVariate function generates n values from which the Mean and Standard Deviation are determined. This approach is similar to the MonteCarlo method recommended in [2].

The *symbolic*, *exact numeric*, and *numeric* results are repeatable. The *statistical* results are slightly different at every evaluation. For large value of n the *statistical* results are very close to the *numeric* results.

Mathematica solve integrals for continuous distributions and sums for discrete distributions. For symbolic and

exact numeric input analytical methods and exact arithmetic are used. For numeric input numerical methods are used. However there are expressions for which only statistical results are applicable.

5. Linear expressions

Example 1 and example 2 give uncertainty of the expression $c_1*x_1+c_2*x_2+c_3*x_3$ when $x_1, x_2,$ and x_3 are independent random variables that follow the normal distributions. The example 3 obtain only the expectation by using the built-in Mathematica function `Expectation`. Surprisingly, this function is much slower than `Uncertainty` function. This is very strange, because `Uncertainty` explore only built-in *Mathematica* functions. The function `Timing` return the time spent in the evaluation and the result obtained. The input lines that needs more than a few seconds start with `Timing`. The pop-up menu input contain “example 3, 20 sec”, which means that the evaluation will spent about 20 seconds. During evaluation the bracket of the sell is marked. *Mathematica* returns a list of the time used, together with the result obtained. Calculator show them in two lines: 20.016 is time in seconds, $c_1 \mu_1 + c_2 \mu_2 + c_3 \mu_3$ is the result. Note that the repeated evaluation are always shorter. The second evaluation spent about 7.391 seconds.

+

```

UncertaintyGUM[c1*x1+c2*x2+c3*x3, {x1==μ1+σ1,x2==μ2+σ2,x3==μ3+σ3},1]
(c1 μ1 + c2 μ2 + c3 μ3) ± √(c12 σ12 + c22 σ22 + c32 σ32)

Uncertainty[c1*x1+c2*x2+c3*x3,
{x1≈NormalDistribution[μ1,σ1],
x2≈NormalDistribution[μ2,σ2],
x3≈NormalDistribution[μ3,σ3]}]
(c1 μ1 + c2 μ2 + c3 μ3) ± √(c12 σ12 + c22 σ22 + c32 σ32)

Timing[Expectation[c1*x1+c2*x2+c3*x3,
{x1≈NormalDistribution[μ1,σ1],
x2≈NormalDistribution[μ2,σ2],
x3≈NormalDistribution[μ3,σ3]}]]]
20.016
c1 μ1 + c2 μ2 + c3 μ3

Timing[Expectation[c1*x1+c2*x2+c3*x3,
{x1≈NormalDistribution[μ1,σ1],
x2≈NormalDistribution[μ2,σ2],
x3≈NormalDistribution[μ3,σ3]}]]]
7.391
c1 μ1 + c2 μ2 + c3 μ3

```

input

▼

clear all

enter

Example 4 and example 5 obtain uncertainty of the expression $c_1*x_1+c_2*x_2+c_3*x_3$ when $x_1, x_2,$ and x_3 are fully correlated random variables that follow the multinormal distributions.

```

UncertaintyGUM[c1*x1+c2*x2+c3*x3,
{x1==μ1±σ1,x2==μ2±σ2,x3==μ3±σ3},3]
(c1 μ1 + c2 μ2 + c3 μ3) ± (c1 σ1 + c2 σ2 + c3 σ3)

PowerExpand[Simplify[Uncertainty[c1*x1+c2*x2+c3*x3,
{x1,x2,x3}≈MultinormalDistribution[{μ1,μ2,μ3},
{{σ1^2,σ1*σ2,σ1*σ3},
{σ1*σ2,σ2^2,σ2*σ3},
{σ1*σ3,σ2*σ3,σ3^2}}]]]]
(c1 μ1 + c2 μ2 + c3 μ3) ± (c1 σ1 + c2 σ2 + c3 σ3)

```

input

Example 6 solve example 4 when the random variables $x_1, x_2,$ and x_3 are correlated by the correlation matrix $\{\{1, r_{12}, r_{13}\}, \{r_{12}, 1, r_{23}\}, \{r_{13}, r_{23}, 1\}\}$, where $-1 \leq r_{i,j} \leq 1$.

Example 7 solve example 5 when the random variables $x_1, x_2,$ and x_3 are correlated by the matrix $\{\{\sigma_1^2, \sigma_1*\sigma_2, \sigma_1*\sigma_3\}, \{\sigma_1*\sigma_2, \sigma_2^2, \sigma_2*\sigma_3\}, \{\sigma_1*\sigma_3, \sigma_2*\sigma_3, \sigma_3^2\}\}$ * $\{\{1, r_{12}, r_{13}\}, \{r_{12}, 1, r_{23}\}, \{r_{13}, r_{23}, 1\}\}$ and follow the multinormal distribution.

+

```

ExpandAll[UncertaintyGUM[c1*x1+c2*x2+c3*x3,
{x1==μ1±σ1,x2==μ2±σ2,x3==μ3±σ3},
{{1,r12,r13},
{r12,1,r23},
{r13,r23,1}}]]
(c1 μ1 + c2 μ2 + c3 μ3) ±
√(c12 σ12 + 2 c1 c2 r12 σ1 σ2 + c22 σ22 + 2 c1 c3 r13 σ1 σ3 + 2 c2 c3 r23 σ2 σ3 + c32 σ32)

Expand[Uncertainty[c1*x1+c2*x2+c3*x3,
{x1,x2,x3}≈MultinormalDistribution[{μ1,μ2,μ3},
{{σ12,σ1*σ2,σ1*σ3},
{σ1*σ2,σ22,σ2 σ3},
{σ1*σ3,σ2*σ3,σ32}}]*
{{1,r12,r13},
{r12,1,r23},
{r13,r23,1}}]]]
(c1 μ1 + c2 μ2 + c3 μ3) ±
√(c12 σ12 + 2 c1 c2 r12 σ1 σ2 + c22 σ22 + 2 c1 c3 r13 σ1 σ3 + 2 c2 c3 r23 σ2 σ3 + c32 σ32)

```

input
▼
clear all
enter

Because the expression $c1*x1+c2*x2+c3*x3$ is linear the functions `UncertaintyGUM` and `Uncertainty` gives the same results in the above examples.

Example 8 find the uncertainty of the same expression $c1*x1+c2*x2+c3*x3$ when the random variables $x1$, $x2$, and $x3$ are independent and follow some of the numerous Mathematica distributions. Only the `Uncertainty` function is able to solve this examples.

+

```

Timing[FullSimplify[Uncertainty[c1*x1+c2*x2+c3*x3,
  {x1 ≈ UniformDistribution[{μ1-σ1,μ1+σ1}],
  x2 ≈ TriangularDistribution[{μ2-σ2,μ2+σ2}],
  x3 ≈ MaxwellDistribution[σ3]}]]]
0.375

$$\left( c_1 \mu_1 + c_2 \mu_2 + 2 c_3 \sqrt{\frac{2}{\pi}} \sigma_3 \right) \pm \sqrt{\frac{c_1^2 \sigma_1^2}{3} + \frac{c_2^2 \sigma_2^2}{6} + \frac{c_3^2 (-8+3\pi) \sigma_3^2}{\pi}}$$


```

input

▼

clear all

enter

6. Non-linear expression: Sin[x]

This section finds the uncertainty of the non-linear expressions Sin[x]. The results obtained by UncertaintyGUM and Uncertainty functions are compared.

Example 9 and Example 10 gives uncertainty of Sin[x] when the random variable x is given by $x = \mu \pm \sigma$, or x follows the normal distribution.

+

```

Simplify[UncertaintyGUM[Sin[x],x == μ±σ, 2]]
Sin[μ] ± √(σ² Cos[μ]² - ¼ σ⁴ (1 + 3 Cos[2 μ]))

```

```

Timing[FullSimplify[Map[ExpToTrig,Uncertainty[Sin[x],
x≈NormalDistribution[μ,σ]]]]/.(a_+Exp[c_])→Exp[c](a/Exp[c]+1)]
85.219

```

$$e^{-\frac{\sigma^2}{2}} \sin[\mu] \pm \frac{\sqrt{(1-e^{-\sigma^2})(1+e^{-\sigma^2} \cos[2\mu])}}{\sqrt{2}}$$

input

▼

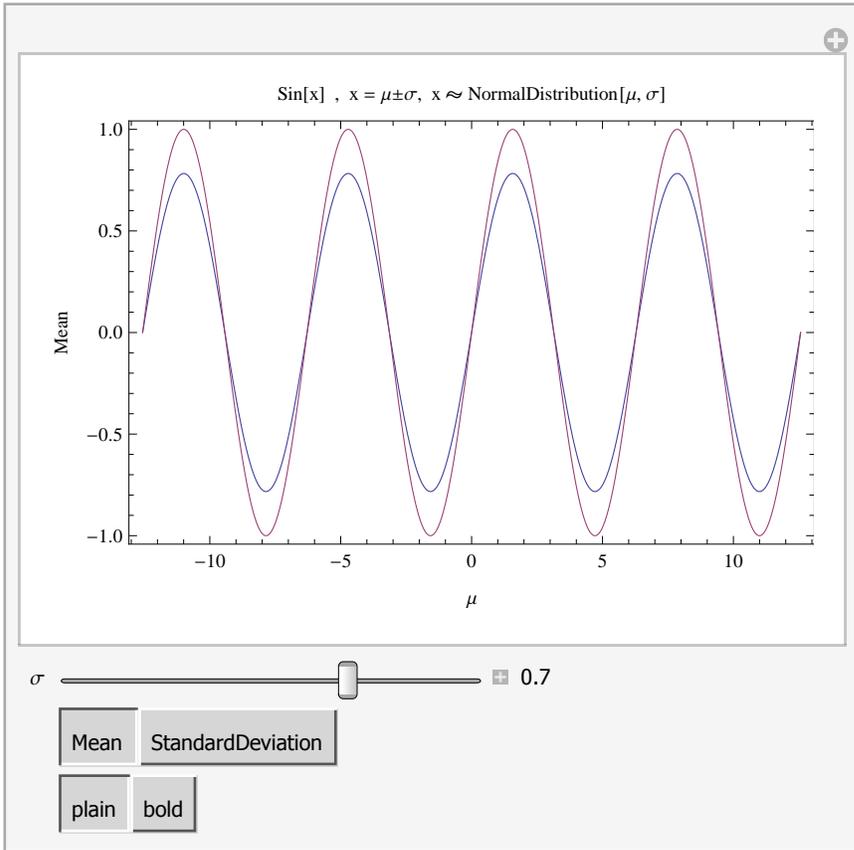
clear all

enter

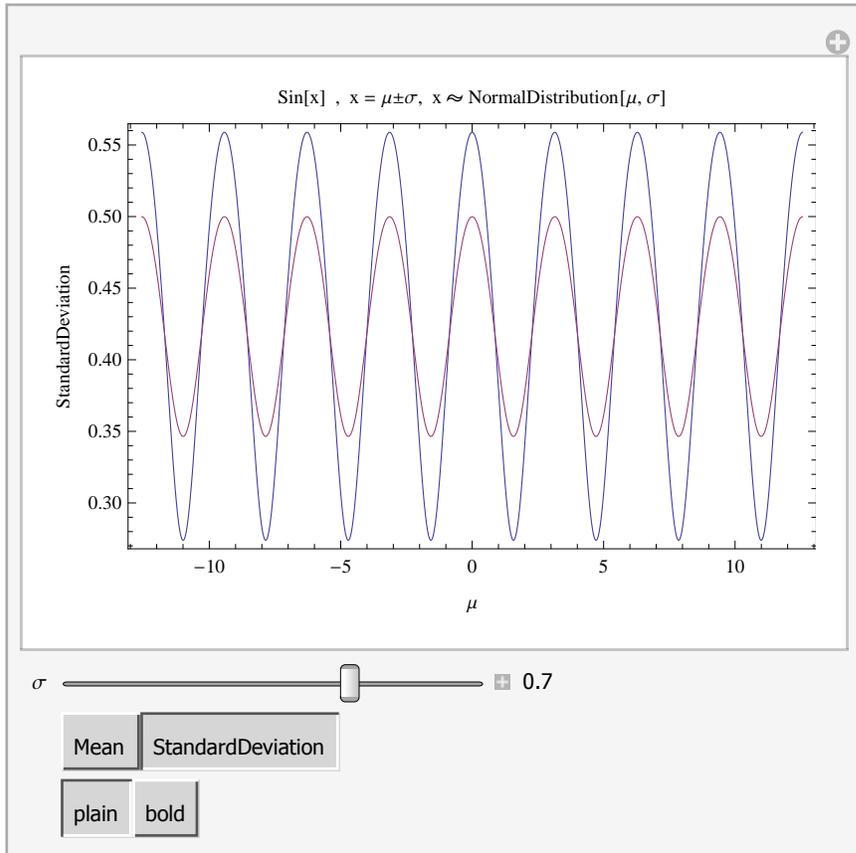
Demonstration 1

This demonstration compare the mean and standard deviation obtained in Example 9 and Example 10. For easy reading and transforming the paper in pdf format, the demonstration is pasted twice to show the mean and standard deviation

plots. The CDF file file with extension .cdf gives both plots after a click on “Mean” or “StandardDeviation” buttons. When the mouse cursor is on red curve “GUM” appears.



Move the slider σ to investigate the differences between the "GUM" and exact solutions. The mean of both solution coincide for $0 \leq \sigma \leq 0.2$. Increasing the values of σ reduce the mean. Since the "GUM" mean is independent of σ the slider σ move only the exact solution (blue curve).



The standard deviation for both solution coincides for $0 \leq \sigma \leq 0.35$. For larger value of σ only "GUM" curve considerably change. For $\sigma = 0.8162$ it becomes a line and for larger σ reverse its periodical behavior. As expected, for small values of σ the "GUM" solution gives correct results.

7. Non-linear expression: x^n

This section finds the uncertainty of the non-linear expressions x^n , where n is integer. The results obtained by `UncertaintyGUM` and `Uncertainty` functions are compared. Example 11 and Example 12 gives uncertainty of x^n when the random variable x is given by $x = \mu \pm \sigma$, or x follows the normal distribution, respectively.

+

```

Timing[FullSimplify[UncertaintyGUM[x^n,x == μ±σ, 2],
n ∈ Integers]]
1.828

$$\mu^n \pm \frac{\sqrt{\mu^{-4+2n} (2 \mu^2 \sigma^2 + (-1+n) (-5+3n) \sigma^4)} \text{Abs}[n]}{\sqrt{2}}$$


Timing[FullSimplify[Uncertainty[x^n,
x≈NormalDistribution[μ,σ]],n ∈ Integers]]
70.875

$$\frac{1}{\sqrt{\pi}}$$


$$2^{-1+\frac{n}{2}} \sigma^{-1+n} \left( -\sqrt{2} (-1 + (-1)^n) \mu \text{Gamma}\left[1 + \frac{n}{2}\right] \text{Hypergeometric1F1}\left[\frac{1}{2} - \frac{n}{2}, \frac{3}{2}, -\frac{\mu^2}{2\sigma^2}\right] + \right.$$


$$\left. (1 + (-1)^n) \sigma \text{Gamma}\left[\frac{1+n}{2}\right] \text{Hypergeometric1F1}\left[-\frac{n}{2}, \frac{1}{2}, -\frac{\mu^2}{2\sigma^2}\right] \right) \pm \frac{1}{\sqrt{\pi}} \left( \sqrt{\left( 2^{-1+n} \sigma^{2n} \right.} \right.$$


$$\left. \left( 1 / \sigma^2 2 (-1 + (-1)^n) \mu^2 \text{Gamma}\left[1 + \frac{n}{2}\right]^2 \text{Hypergeometric1F1}\left[\frac{1}{2} - \frac{n}{2}, \frac{3}{2}, -\frac{\mu^2}{2\sigma^2}\right]^2 + \right.} \right.$$


$$\left. \left. 2 \sqrt{\pi} \text{Gamma}\left[\frac{1}{2} + n\right] \text{Hypergeometric1F1}\left[-n, \frac{1}{2}, -\frac{\mu^2}{2\sigma^2}\right] - \right.} \right.$$


$$\left. \left. (1 + (-1)^n) \text{Gamma}\left[\frac{1+n}{2}\right]^2 \text{Hypergeometric1F1}\left[-\frac{n}{2}, \frac{1}{2}, -\frac{\mu^2}{2\sigma^2}\right]^2 \right) \right)$$


```

input ▼ clear all enter

The last solution is used to define powerXn function that directly compute the uncertainty of x^n for any specified positive integer. The Example 13 verify that powerXn function gives directly the same results as Uncertainty function for n from 1 to 20.

+

```

?powerXn
powerXn[μ,σ,n] implement the solution of Uncertainty[x^n,
x≈NormalDistribution[μ,σ]], n ∈ Integers]

```

input ▼ clear all enter

```

?powerXn
powerXn[μ,σ,n] implement the solution of Uncertainty[x^n,
x≈NormalDistribution[μ,σ]], n ∈ Integers]

Timing[And@@PowerExpand[Map[Simplify[powerXn[μ,σ,#]==
  Uncertainty[x^#,x≈NormalDistribution[μ,σ]]&,Range[20]]]]]
1.156
True

```

input

Example 14 and example 15 obtain uncertainty given by `UncertaintyGUM` and `Uncertainty` for x^2 and $x = \mu \pm \sigma$ or $x \approx \text{NormalDistribution}[\mu, \sigma]$, respectively. For symbolic inputs we obtain different expectation μ^2 and $\mu^2 + \sigma^2$, but the same uncertainty.

```

UncertaintyGUM[x^2,x == μ±σ, 2]

$$\mu^2 \pm \sqrt{4 \mu^2 \sigma^2 + 2 \sigma^4}$$


Uncertainty[x^2,x≈NormalDistribution[μ,σ]]

$$(\mu^2 + \sigma^2) \pm \sqrt{2} \sqrt{2 \mu^2 \sigma^2 + \sigma^4}$$


```

input

Example 16 shows that input of exact numbers (integers or rationals) gives exact result. Note that such number could be evaluated with arbitrary number of digits after decimal point. The result in Example 17 is reals since the input include the real number 0.1. The results are repeatable. Any evaluation give the same result.

+

```

Uncertainty[x^2,x~NormalDistribution[10,1/10]]

```

$$\frac{10001}{100} \pm \frac{\sqrt{\frac{20001}{2}}}{50}$$

```

Uncertainty[x^2,x~NormalDistribution[10,.1]]
100.01 ± 2.00005

Uncertainty[x^2,x~NormalDistribution[10,.1]]
100.01 ± 2.00005

```

input

▼

clear all

enter

The example 18 generates one million random variates. After every evaluation the statistical result is slightly different, but very close to the numerical one, since 1 million random variate are used. This method is improved version of Monte Carlo method. Its advantage is that it is applicable for any problems.

+

```

Uncertainty[x^2,x~NormalDistribution[10,.1],10^6]
100.01 ± 2.00235

Uncertainty[x^2,x~NormalDistribution[10,.1],10^6]
100.01 ± 1.99733

Uncertainty[x^2,x~NormalDistribution[10,.1],10^6]
100.009 ± 1.99956

```

input

▼

clear all

enter

Demonstration 2

This demonstration gives the analytical results of `UncertaintyGUM` and `Uncertainty` for x^n . Click on different n to see the difference between the results.

+

UncertaintyGUM[x^3,x == μ±σ, 2]

$$\mu^3 \pm \sqrt{9 \mu^4 \sigma^2 + 36 \mu^2 \sigma^4}$$

Uncertainty[x^3,x ≈ NormalDistribution[μ,σ]

$$(\mu^3 + 3 \mu \sigma^2) \pm \sqrt{9 \mu^4 \sigma^2 + 36 \mu^2 \sigma^4 + 15 \sigma^6}$$

n

1	2	3	4	5	6	7	8	9	10
---	---	---	---	---	---	---	---	---	----

Demonstration 3

This demonstration gives the numerical results of `UncertaintyGUM` and `Uncertainty` for x^n . Change the value of μ , and σ . Increase the number of random variate n to see that statistical results become very close to the correct results.

+

UncertaintyGUM[x^1,x==1±0.02,2]

1 ± 0.02

Uncertainty[x^1,x≈NormalDistribution[1,0.02]

1 ± 0.02

Uncertainty[x^1,x≈NormalDistribution[1,0.02,10]

0.99823 ± 0.0285548

n

1	2	3	4	5	6	7	8	9	10
---	---	---	---	---	---	---	---	---	----

μ ± 1

σ ± 0.02

The number of RandomVariates:

10	10 ²	10 ³	10 ⁴	10 ⁵	10 ⁶
----	-----------------	-----------------	-----------------	-----------------	-----------------

■ **7. Conclusion**

The `Uncertainty` function is more powerful than `UncertaintyGUM` function, because it could obtain symbolically, exactly, numerically, or statistically the expectation ± standard deviation for expression of variables with prescribed statistical distributions. The mean and standard deviation of `Sin[x]` obtained approximately by `Uncer-`

taintyGUM and exactly by Uncertainty functions coincide for $0 \leq \sigma \leq 0.2$ and standard deviation for $0 \leq \sigma \leq 0.35$. As expected, for small values of σ the “GUM” solution gives correct results. The `powerXn` function, based on the exact solution obtained in Example 12 demonstrate the possibility to create new functions for the frequently used nonlinear models. The statistical solution using `RandomVariate` are similar to the MonteCarlo method recommended in [2].

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